

Aspects of the Coarse-Grained-Based Approach to a Low-Relativistic Fractional Schrödinger Equation

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Abstract

The main goal of this paper is to set up the coarse-grained formulation of a fractional Schrödinger equation that incorporates a higher (spatial) derivative term which accounts for relativistic effects at a lowest order. The corresponding continuity equation is worked out and we also identify the contribution of the relativistic correction the quantum potential in the coarse-grained treatment. As a consequence, in the classical regime, we derive the sort of fractional Newtonian law with the quantum potential included and the fractional counterparts of the De Broglies's energy and momentum relations.

1 Introduction

Physicists are presently seeking and trying to understand the connections between complex systems, nonlocal field theories and other areas of Physics. This is today an important subject of studies in different physical and mathematical areas, but the understanding of non-linear processes connected to these topics has had a considerable boost over the past 40 years. This deeper comprehension has been inspired by the discovery and the insight of a new phenomenon, known as dynamical chaos. The main motivation is that the use of these theories may yield a much more elegant and effective treatment of problems in particle and high energy physics, as it has hitherto been carried out with the help of the local field theories.

A particular subclass of non-local field theories is described with the operators of a fractional nature and is specified in the framework of fractional calculus (FC). FC provides us with a set of mathematical tools to generalize the concept of derivative and integral operators with integer order to their respective extensions of an arbitrary real order. FC has raised up a great deal of interest over recent years and has been used as an applied tool to the study of fractional dynamics in many fields of physics, mechanics, engineering and other areas to approach problems connected with complex systems [1]. Today, there is a rich stream of works linking such areas throughout different paths, [2]. Non-local theories and memory effects can also be connected to complexity and admit a treatment in terms of FC. In this context, the non-differentiable nature of the microscopic dynamics may be connected with time scales so as to approach questions in the realm of complex systems [3].

The inclusion of relativistic effects into the Schrödinger equation intend to the correct computation of the atomic spectrum and in the area of heavy-ion collisions, relativistic contributions are typically much larger and, especially for atoms with large nuclear charges Z , these effects can be quite significant [4]. Relativistic contributions have also been considered in the studies of the electron motion in the operation of free-electron lasers. Various important characteristics of a quantum system may not be well determined if those relativistic effects are not completely taken into account in calculation, e.g in the context of the atomic and molecular structure and the energy levels, in the describing of excited states and the fine structure of an hydrogen like atoms and so on.

According to K. G. Dyall [5], experimental evidences with atomic structure has shown that calculations which proceed from a fully relativistic model are longer than the corresponding nonrelativistic calculations. To ascertain what level of accuracy is required, we have to make the inclusion of relativistic effects that themselves makes a significant difference to the results, or whether the error in including them by perturbation theory is significant. Neglecting this picture change may lead to serious inaccuracies, e.g., the calculations for order-of-magnitude estimates of a quantity may not need to consider relativity at all, whereas calculations of experimental accuracy on the hydrogen atom must account for relativistic effects in detail. In making an assessment of whether relativistic effects should be included, and by what means, it is useful to have an overview of the effect of relativity on structure, the size of relativistic effects in the periodic system and some criteria for judging their importance. Also, following M. Reither and B. Heß [6], this description are also important for the interpretation of highly accurate experiments in spectroscopy. As already mentioned, the so-called relativistic effects begin to play a major role in heavy atoms and their compounds. This is due to the fact that the relativistic effects on energies and other physical quantities increase with the fourth power of the nuclear charge Z .

The relativistic linear Schrödinger equation has been discussed at the early years of quantum mechanics but was dismissed promptly by the Klein-Gordon and the Dirac equations. Recently, relativistic versions of the Schrödinger equation have been considered in the study of relativistic quark-anti-quark bound states [7], and gravitational collapse of a boson star [8].

In the present, we pursue an investigation of the coarse-grained fractional Schrödinger equation corrected by fourth spatial derivative term which accounts for a relativistic correction to the kinetic energy term in the Hamiltonian.

Recently, Guy Jumarie [9] proposed an alternative definition to the Riemann-Liouville derivative. His modified

Riemann-Liouville derivative (MRL) has the advantages of both the standard Riemann-Liouville and Caputo fractional derivatives: it is defined for arbitrary continuous (non-differentiable) functions and the fractional derivative of a constant is equal to zero. The MRL approach seems to give a mathematical framework for dealing with dynamical systems defined on coarse-grained spaces and with coarse-grained time and, to this end, to use the fact that fractional calculus appears to be intimately related to fractal and self-similar functions. The well-tested definitions for fractional derivatives, namely, Riemann-Liouville and Caputo have been frequently used for several applications. In spite their adequacy, they have some dangerous pitfalls. For this reason we use here the MRL approach of fractional derivative. Basic definitions [9, 3, 10, 11, 12, 13] and detail of the formalism can be found in the cited references and references therein.

We would like to emphasize that the choice of MRL approach, besides the points already mentioned, is justified by the fact that the chain and Leibniz rules acquires a simpler form, which helps a great deal if changes of coordinates are performed. Moreover, causality seems to be more easily obeyed in a field-theoretical construction if we adopt this approach.

In a previous work [3], we have argued that the modelling of TeV-physics may demand an approach based on fractal operators and FC. We claimed that, in the realm of complexity, non-local theories and memory effects were connected to complexity and the FC and that the non-differentiable nature of the microscopic dynamics may be connected with time scales. Using the MRL definition of fractional derivatives, we have worked out explicit solutions to a fractional wave equation with suitable initial conditions to carefully understand the time evolution of classical fields with a fractional dynamics. First, by considering space-time partial fractional derivatives of the same order in time and space, a generalized fractional D'Alembertian is introduced and by means of a transformation of variables to light-cone coordinates, an explicit analytical solution were obtained. Also, aspects connected with Lorentz symmetry were analyzed in two different approaches.

Here, we claim that the use of an approach based on a sequential form of MRL [9] is more appropriate to describe the dynamics associated with field theory and particle physics in the space of non-differentiable solution functions, or in the coarse-grained space-time. Based on this approach, we have worked out a suggested version of a fractional Schrödinger equation, with a lowest-order relativistic correction, obtained starting from a fractional wave equation [3] to which a mass term has been adjoined, to give us a fractional Klein Gordon equation (FKGE), and also with the help of the definition of some fractional operators and McLaurin expansion. By a plane wave ansatz of solutions, we have obtained fractional versions of Bohmian equations to describe the particle dynamics associated with Bohmian mechanics theory and physics, in the space of non-differentiable solution functions to the referred fractional Schrödinger equation with the lowest-order relativistic correction.

As pointed out by Jumarie, non-differentiability and randomness are mutually related in their nature, in such a way that studies in fractals on the one hand and fractional Brownian motion on the other hand are often parallel in the same paper. A function which is continuous everywhere but is nowhere differentiable necessarily exhibits random-like or pseudo-random-features, in the sense that various samplings of this functions on the same given interval will be different. This may explain the huge amount of literature which extends the theory of stochastic differential equation to stochastic

dynamics driven by fractional Brownian motion.

The most natural and direct way to question the classical framework of physics is to remark that in the space of our real world, the generic point is not infinitely small (or thin) but rather has a thickness. A coarse-grained space is a space in which a generic point is not infinitely thin, but rather has a thickness; and here this feature is modelled as a space in which the generic differential is not dx , but rather $(dx)^\alpha$ and likewise for coarse grained with respect to the time variable t . It is noteworthy, at this stage, to highlight the interesting work by Nottale [14], where the notion of fractal space-time is first introduced.

In our work, the most important rules in the MRL definition used here is that the derivative of constant is zero, we can use it so much for differentiable as non differentiable functions, it has simple chain and Leibniz rules that are similar to integer derivatives.

Our paper is outlined as follows: In Section 2, we review the development of the low-relativistic correction to the integer order Schrödinger equation and discuss the fractional Klein Gordon equation. In Section 3, we develop the low-relativistic fractional Schrödinger equation and present the fractional continuity equation. In Section 4 we work out the fractional continuity equation. Section 5 is devoted to the development of the fractional Bohmian equations with low-relativist limit. Finally, in Section 6 we cast our the concluding comments and prospects for further investigation.

2 Lowest-Order Relativistic Corrections to the Integer Schrödinger Equation and the Fractional Klein Gordon Equation

If we start off from the well-known relativistic relation

$$E = \sqrt{\vec{p}^2 c^2 + m^2 c^4}, \quad (1)$$

$$\begin{cases} E = m_0 c^2 \gamma; \\ \vec{p} = m_0 \vec{v} \gamma \end{cases} \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \quad (2)$$

We readily get that $\frac{\vec{p}c}{E} = \frac{\vec{v}}{c}$. So, in the non-relativistic regime ($|\vec{v}| \ll c$), $|\vec{p}|c \ll E$ and so the following approximation can be adopted:

$$\begin{aligned} E - mc^2 &\equiv \varepsilon_{nr} \cong \frac{p^2}{2m} - \frac{p^4}{8m^3 c^2} = \\ &= \frac{(pc)^2}{2mc^2} \left[1 - \left(\frac{pc}{2mc^2} \right)^2 \right], \end{aligned} \quad (3)$$

where ε_{nr} stands for the non-relativistic kinetic energy. Here, it is worthy of notice that the lowest-order relativistic limit corresponds to momenta such that other terms $|\vec{p}|c \gg 2mc^2$, that is the threshold energy for a pair creation. The fractional approach here is still justified by the argumentation that the particle described by this formalism is actually a

pseudo-particle that carries the information of the media and the kind interaction implicit in the equation that describes his evolution. This pseudo-particle is then “dressed” with information about media and interactions, and the solutions of the fractional equation are, like the Green functions in condensed matter physics, carrying additional information about iterations and media. Then, even if the media is not fractal, due to not so high energy regime, the fractional approach still makes sense to describe the evolutions of a pseudo-particle. This means that essentially there is not an isolated particle in the fractional approach context but an pseudo-particle “dressed” with information about the fields and interactions in the media.

Now defining the quantum mechanics one dimensional operators, energy and linear momentum, as usual

$$\begin{cases} \hat{E} = i\hbar \frac{\partial}{\partial t} \\ \hat{p} = -i\hbar \frac{\partial}{\partial x} \end{cases} ; \quad (4)$$

we will obtain the Schrödinger with lowest-order relativistic correction that reads,

$$i\hbar \frac{\partial}{\partial t} \psi(x, t) = - \left(\frac{\hbar^2}{2m} \right) \frac{\partial^2}{\partial x^2} \psi(x, t) - \left(\frac{\hbar^4}{8m^3 c^2} \right) \frac{\partial^4}{\partial x^4} \psi(x, t) + V \psi(x, t). \quad (5)$$

With the lowest-order relativistic correction to Schrödinger equation, we then construct in the sequence the fractional Klein Gordon equation by adjoined a mass term to the fractional wave equation.

The Fractional Klein Gordon Equation

In a recent paper [3], we have obtained in a natural way the fractional wave equation.

Now, we shall write down a fractional version of the Klein Gordon equation by the addition of the mass term to the fractional wave equation, considering adequate dimension scale factors, in order to gain some insight to about the fractional quantum operator to be used.

The usual KG equation reads

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \psi(x, t) - \frac{\partial^2}{\partial x^2} \psi(x, t) + \frac{m^2 c^2}{\hbar^2} \psi(x, t) = 0. \quad (6)$$

Fractional Klein-Gordon equation [15] and fractional Dirac equation have been studied by several authors over the past decade [16, 17, 18]. Some articles have been dealing with fractional power of D’Alembertian operator used in the non local kinetic terms Lagrangian field theory in the (2+1)-dimensional bosonization and also to study the effective field theory, which has some degrees of freedom integrated out from the underlying local theory [19, 20, 21]. The canonical quantization of fractional massless and massive fields has been studied by some authors [22, 23] and quantization of fractional Klein-Gordon field and fractional gauge field based on Nelson’s stochastic mechanics and Parisi-Wu stochastic quantization procedure at zero and positive temperature have been considered [24, 25]. An axiomatic approach to fractional Klein-Gordon field, where properties of the n-point Schwinger or Euclidean Green functions and their analytic continuation to the corresponding n-point Wightman functions were studied by [26, 27].

The fractional KG equation can be written here, in an similar manner as in ref. [28], but with different fractional orders in space and time, as

$$\frac{1}{c^{2\beta}} \frac{\partial^{2\beta}}{\partial t^{2\beta}} \psi(x, t) - M_{x,\alpha}^2 \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} \psi(x, t) + \frac{m^{2\beta} c^{2\beta}}{\hbar^{2\beta}} \psi(x, t) = 0. \quad (7)$$

The diffusion factor $M_{x,\alpha}$ is here introduced for dimensional consistency reasons. This equation has also to be consistent with an fractional relativistic energy-momentum equation, given by

$$E_\beta = \sqrt{p^{2\alpha} c^{2\alpha} + m^{2\beta} c^{4\beta}}. \quad (8)$$

Now, with these considerations, we shall expand the momentum energy of eq.(8) in terms of an integer McLaurin series and, after the substitution of fractional quantum operators, obtain the fractional Schrödinger equation with lowest-order relativistic correction term.

3 Fractional Schrödinger Equation with Lowest-Order Relativistic Correction

A method first used for the attainment of a fractional Schrödinger equation was the path integral over the Lévy paths formalisms [29, 30] where a fractional generalization of the Schrödinger equation in terms of the quantum Riesz fractional derivative was obtained and there have been analyzed the energy spectra of a hydrogen-like atom and of a fractional oscillator in the semi-classical approximation and the parity conservation law. The argumentation to achieve equation [31] was that a the path integral over Brownian trajectories leads to the well-known Schrödinger equation, then the path integral over Lévy trajectories leads to the space fractional Schrödinger equation. Other versions of Schrödinger equation were obtained [32] considering only a time fractional Schrödinger equation in the sense of a Caputo fractional time derivative formalism. A version of generalized fractional Schrödinger, with space-time fractional derivatives in the sense of Caputo and Riesz fractional derivatives, was studied in ref. [33] and solved for free particle and square well potential with integral transform methods. The fractional Schrödinger equation can also be obtained by methods like a fractional variational method in the context of a Lagrangian formulation or by a fractional Klein Gordon equation [28].

Here we adopt the MRL approach for fractional derivatives that is less restrictive than other definitions, to obtain the lowest-order relativistic correction to a fractional Schrödinger equation, with different orders for the fractional derivatives in time and space, by means of a fractional Klein Gordon equation that, by other way, was obtained from a fractional wave equation in our recent work [3].

The main rules used here in the MRL approach are summarized as

$D^\alpha K = 0$, K is constant, $D^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha}$, $\gamma > 0$, derivative of power function, $(u(x)v(x))^{(\alpha)} = u^{(\alpha)}(x)v(x) + u(x)v^{(\alpha)}(x)$ is the Leibniz rule. The chain rule for non differentiable functions is written as

$$\frac{d^\alpha}{dx^\alpha} f[u(x)] = \frac{d^\alpha}{du^\alpha} f \left(\frac{d}{dx} u \right)^\alpha, \quad (9)$$

where f is α -differentiable and u is differentiable with respect to x and, for coarse-grained space-time as

$$\frac{d^\alpha}{dx^\alpha} f[u(x)] = \frac{d}{du} f \frac{d^\alpha}{dx^\alpha} u, \quad (10)$$

where $f(u(x))$ is not differentiable w.r.t x but it is differentiable w.r.t u , and u is not differentiable w.r.t x .

For further details, the readers can follow the refs. [11, 12] which contain all the basic for the formulation of a fractional differential geometry in coarse-grained space, and refers to an extensive use of coarse-grained phenomenon.

It is worthy to point out that the Leibniz rule used here is a good approximation that comes from the first two terms of the fractional Taylor series development, that holds only for nondifferentiable functions [12] and are as good and approximated as the classical integer one. Here, a comment is pertinent: the fractional MRL approach for non-differentiable functions has similar rules and has definition with a mathematical limit operation comparable to certain definitions of local fractional derivatives, as that introduced by Kolwankar and Gandal [34, 35, 36] with some studies in the literature. For example, the works of Refs. [37, 38, 39] or the approaches with Hausdorff derivative, also called fractal derivative [40, 41], that can be applied to power-law phenomena and the recently developed α -derivative [42]. The MRL approach seems to us to be an integral version of the calculus mentioned above and all of them deserve to be more deeply investigated, under a mathematical point of view, in order to give exact differences and similarities respect to the traditional fractional calculus with Riemann-Liouville or Caputo definition and with local fractional calculus and even fractional q-calculus [43, 1, 44, 45], as well as in the comparative point of view of physics [41, 44, 45, 46, 47], for the scope of applicabilities.

We think that the referred alternative formalisms can be used to the attainment of results similar or with similarities to some of those here obtained [48] and this is a good indication that our results are more general and not only dependent and provided by an specific formalism.

In this work, we construct the fractional Schrödinger equation based on operator proposed in view of the fractional Klein Gordon equation.

Developing the eq.(8) in McLaurin's series, doing $f(x) = (1 + x_{\alpha,\beta})^{1/2}$ and assuming that $f^{(\alpha k)}(x)$ have sequential character like

$$f^{(2\alpha)}(x) = \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} = \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\alpha}{\partial x^\alpha}, \quad (11)$$

$$f(x) = (1 + x_{\alpha,\beta})^{1/2} \cong 1 + \frac{1}{2}x_{\alpha,\beta} - \frac{1}{8}x_{\alpha,\beta}^2 + \mathcal{O}(x_{\alpha,\beta}^3). \quad (12)$$

Since the semi group properties for fractional derivatives does not hold in general, we used the Miller-Ross sequential derivative [49] in the MRL sense. Incidentally, the Miller-Ross sequential derivative is a systematic procedure that carries out a fractional higher-order derivative while avoiding the recursive application of many single derivatives taken after each other. Moreover, we took the option to carry out the sequence of derivatives in the cascade form, in MRL sense, as done in the work of ref. [13, 12].

Here, we propose the operators:

$$\begin{cases} \widehat{E}_\beta = i (\hbar)^\beta \frac{\partial^\beta}{\partial t^\beta} \\ \widehat{p}_\alpha = -i (\hbar)^\alpha M_{x,\alpha} \frac{\partial^\alpha}{\partial x^\alpha} \end{cases} ; \quad (13)$$

It can be verified that the fractional quantum operator proposed above, when substituted into the equation eq.(8) will give the KG equation eq.(7).

Note that $M_{x,\alpha}$ factor becomes dimensionless when α equal to 1 and its mass dimension is in general α -dependent.

Now, evidencing the term $m^\beta c^{2\beta}$ in the eq.(8) and expanding in terms of McLaurin's series in $x_{\alpha,\beta} = \frac{p^{2\alpha} c^{2\alpha}}{m^{2\beta} c^{4\beta}}$, by substitution of the fractional operators in eq. (13), we are lead to one possible representation of the fractional Schrödinger equation given by

$$\begin{aligned} i (\hbar)^\beta \frac{\partial^\beta}{\partial t^\beta} \psi(x, t) = & -M_{x,\alpha}^2 \frac{\hbar^{2\alpha}}{2m^\beta} \frac{c^{2\alpha}}{c^{2\beta}} \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} \psi(x, t) + V_{\alpha,\beta} \psi(x, t) + \\ & - \frac{1}{8} M_{x,\alpha}^4 \frac{\hbar^{4\alpha}}{m^{3\beta}} \frac{c^{4\alpha}}{c^{6\beta}} \frac{\partial^{4\alpha}}{\partial x^{4\alpha}} \psi(x, t) \end{aligned} \quad (14)$$

where the notation is assumed $\frac{\partial^{4\alpha}}{\partial x^{4\alpha}} = \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\alpha}{\partial x^\alpha}$, $\frac{\partial^{2\alpha}}{\partial x^{2\alpha}} = \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\alpha}{\partial x^\alpha}$, since the semi-group properties for additive in the orders of the derivatives may not hold, as previously commented.

4 Fractional Continuity Equation

It is well-known that in standard quantum mechanics continuity equation has a very important figure, it represents a conservation law. In the context of standard quantum mechanics, in the Copenhagen he, we are lead to the conservation of the probability density. But, in the context of a fractional quantum mechanics, the meaning of a fractional continuity equation is not quite clear and require some analysis. Since we are in the interaction picture and are handling with pseudo-particles or "dressed" particles, the fractional continuity equation could give us in true, the revelation that it exist a dissipation implicit in the fractional evolution equations, specially if the orders of derivatives in space and time were different from each other. This could mean that the fractional equations can be thought of related to some effective theories. The known and unknown information about interactions and the media could be accounted for in fractionality. When the integer order limit for derivatives are reached, the conservation law emerges, the dissipation are no more present in the theory and certain symmetries could be reestablished.

We expect that future scientific investigations may clarify more the real meaning of the fractional continuity equation.

To obtain our fractional continuity equation we now proceed as follows: the conjugate of fractional Schrödinger equation reads

$$\begin{aligned} -i (\hbar)^\beta \frac{\partial^\beta}{\partial t^\beta} \psi(x, t) = & -M_{x,\alpha}^2 \frac{\hbar^{2\alpha}}{2m^\beta} \frac{c^{2\alpha}}{c^{2\beta}} \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} \psi(x, t) + V_{\alpha,\beta} \psi(x, t) + \\ & - \frac{1}{8} M_{x,\alpha}^4 \frac{\hbar^{4\alpha}}{m^{3\beta}} \frac{c^{4\alpha}}{c^{6\beta}} \frac{\partial^{4\alpha}}{\partial x^{4\alpha}} \psi(x, t) \end{aligned} \quad (15)$$

Defining the probability as $P = \psi^*(x, t) \psi(x, t)$.

Multiplying (14) by $\psi^*(x, t)$ and equation (15) by $-\psi(x, t)$, after adding both equations, we obtain

$$i(\hbar)^\beta \frac{\partial^\beta}{\partial t^\beta} \psi^*(x, t) \psi(x, t) = -M_{x,\alpha}^2 \frac{\hbar^{2\alpha}}{2m^\beta} \frac{c^{2\alpha}}{c^{2\beta}} \left[\psi^*(x, t) \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} \psi(x, t) - \psi(x, t) \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} \psi^*(x, t) \right] + \\ - \frac{1}{8} M_{x,\alpha}^4 \frac{\hbar^{4\alpha}}{m^{3\beta}} \frac{c^{4\alpha}}{c^{6\beta}} \left[\psi^*(x, t) \frac{\partial^{4\alpha}}{\partial x^{4\alpha}} \psi(x, t) - \psi(x, t) \frac{\partial^{4\alpha}}{\partial x^{4\alpha}} \psi^*(x, t) \right] \quad (16)$$

After some algebra, the latter equation can be written as

$$\frac{\partial^\beta}{\partial t^\beta} \rho(x, t) + \frac{\partial^\alpha}{\partial x^\alpha} J(x, t) = 0, \quad (17)$$

where $\rho(x, t) \equiv \psi^*(x, t) \psi(x, t)$, and

$$J = M_{x,\alpha}^2 \frac{\hbar^{2\alpha}}{2m^\beta i \hbar^\beta} \frac{c^{2\alpha}}{c^{2\beta}} J' + \\ + \frac{1}{8} M_{x,\alpha}^4 \frac{\hbar^{4\alpha}}{m^{3\beta} i \hbar^\beta} \frac{c^{4\alpha}}{c^{6\beta}} \frac{\partial^\alpha}{\partial x^\alpha} \left\{ \left[J' - 2 \left(\frac{\partial^\alpha \psi^*(x, t)}{\partial x^\alpha} \frac{\partial^\alpha \psi(x, t)}{\partial x^\alpha} \right) \right] + 4 \left(\frac{\partial^\alpha \psi(x, t)}{\partial x^\alpha} \frac{\partial^{2\alpha} \psi^*(x, t)}{\partial x^{2\alpha}} \right) \right\}, \quad (18)$$

with $J' \equiv \left[\psi^*(x, t) \frac{\partial^\alpha}{\partial x^\alpha} \psi(x, t) - \psi(x, t) \frac{\partial^\alpha}{\partial x^\alpha} \psi^*(x, t) \right]$.

The equation (17) shows that the probability is conserved in the fractional sense.

Taking $\alpha = \beta = 1$, we obtain the integer continuity equation with the lowest-order relativistic correction.

5 Fractional Quantum Potential with Lowest-Order Relativistic Correction terms

Now, we shall build up the fractional Bohmian equations, by parametrizing the solution of eq.(14) as below:

$$\Psi(\mathbf{r}, t) = R(\mathbf{r}, t) e^{iS(\mathbf{r}, t)/\hbar}, \quad (19)$$

where R and S are the amplitude of probability density and phase of Ψ , respectively, both being real-valued functions. Substituting this relation into the fractional Schrödinger's equation and multiplying by $e^{-iS(\mathbf{r}, t)/\hbar}$, after some algebra and taking real and imaginary parts, we get two equations that lead to a fractional version of Bohmian Mechanics, including the its lowest-order relativistic correction limit.

Now, proceeding as described above, two equations are obtained:

a) for the real part:

$$\begin{aligned}
& -M_{x,\alpha}^2 \frac{\hbar^{2\alpha}}{2m^\beta} \frac{c^{2\alpha}}{c^{2\beta}} \frac{1}{R(x,t)} \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\alpha}{\partial x^\alpha} R(x,t) + M_{x,\alpha}^2 \frac{\hbar^{2\alpha}}{2m^\beta} \frac{c^{2\alpha}}{c^{2\beta}} \frac{1}{\hbar^2} \left(\frac{\partial^\alpha S}{\partial x^\alpha} \right)^2 + \hbar^{\beta-1} \frac{\partial^\beta}{\partial t^\beta} S(x,t) + V + \\
& -\frac{1}{8} M_{x,\alpha}^4 \frac{\hbar^{4\alpha}}{m^{3\beta}} \frac{c^{4\alpha}}{c^{6\beta}} \frac{1}{R} \left[\frac{1}{\hbar^4} R(S^{(\alpha)})^4 + R^{(4\alpha)} - \frac{4}{\hbar^2} R S^{(\alpha)} S^{(3\alpha)} - \frac{12}{\hbar^2} R^{(\alpha)} S^{(\alpha)} S^{(2\alpha)} + -\frac{3}{\hbar^2} R^2 (S^{(2\alpha)})^2 - \frac{6}{\hbar^2} R^{(2\alpha)} (S^{(\alpha)})^2 \right] = 0,
\end{aligned}
\tag{20}$$

b) for the imaginary part:

$$\begin{aligned} & \frac{\partial^\beta R^2}{\partial t^\beta} + 2M_{x,\alpha}^2 \frac{1}{2m^\beta} \frac{c^{2\alpha}}{c^{2\beta}} \hbar^{2\alpha-\beta-1} \frac{\partial^\alpha}{\partial x^\alpha} \left(R^2 \frac{\partial^\alpha S}{\partial x^\alpha} \right) + \\ & - \frac{1}{8} M_{x,\alpha}^4 \frac{\hbar^{4\alpha}}{m^{3\beta}} \frac{c^{4\alpha}}{c^{6\beta}} \frac{i}{\hbar^4} \left(\frac{-2R}{\hbar^\beta} \right) \left[R \hbar^3 S^{(4\alpha)} - 4\hbar R^{(\alpha)} (S^{(\alpha)})^3 + 4\hbar^3 R^{(\alpha)} S^{(3\alpha)} + 6\hbar^3 R^{(2\alpha)} S^{(2\alpha)} \right] = 0. \end{aligned} \quad (21)$$

The first term in the left-hand side of eq.(20) can be called fractional quantum potential by the presence of Planck constant and fractional derivatives

$$\begin{aligned} Q_\alpha(x, t) \equiv & -M_{x,\alpha}^2 \frac{\hbar^{2\alpha}}{2m^\beta} \frac{c^{2\alpha}}{c^{2\beta}} \frac{1}{R(x, t)} \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\alpha}{\partial x^\alpha} R(x, t) + \\ & - \frac{1}{8} M_{x,\alpha}^4 \frac{\hbar^{4\alpha}}{m^{3\beta}} \frac{c^{4\alpha}}{c^{6\beta}} \frac{1}{R} \left[R^{(4\alpha)} - \frac{4}{\hbar^2} R S^{(\alpha)} S^{(3\alpha)} - \frac{12}{\hbar^2} R^{(\alpha)} S^{(\alpha)} S^{(2\alpha)} + -\frac{3}{\hbar^2} R^2 (S^{(2\alpha)})^2 - \frac{6}{\hbar^2} R^{(2\alpha)} (S^{(\alpha)})^2 \right] \end{aligned} \quad (22)$$

With this definition, the eq. (20) can be rewritten as

$$Q_\alpha(x, t) + V + M_{x,\alpha}^2 \frac{\hbar^{2\alpha}}{2m^\beta} \frac{c^{2\alpha}}{c^{2\beta}} \frac{1}{\hbar^2} \left(\frac{\partial^\alpha S}{\partial x^\alpha} \right)^2 + -\frac{1}{8} M_{x,\alpha}^4 \frac{\hbar^{4\alpha}}{m^{3\beta}} \frac{c^{4\alpha}}{c^{6\beta}} \frac{1}{\hbar^4} (S^{(\alpha)})^4 = -\hbar^{\beta-1} \frac{\partial^\beta}{\partial t^\beta} S(x, t), \quad (23)$$

deriving this equation with respect to x^α , interchanging spatial and time ordering of derivatives and considering both fractional derivative orders equals, that is, $\alpha = \beta$, we obtain

$$-\frac{\partial^\alpha}{\partial x^\alpha} (Q_\alpha(x, t) + V) = \frac{\partial^\alpha}{\partial x^\alpha} \left[M_{x,\alpha}^2 \frac{\hbar^{2\alpha}}{2m^\beta} \frac{c^{2\alpha}}{c^{2\beta}} \frac{1}{\hbar^2} \left(\frac{\partial^\alpha S}{\partial x^\alpha} \right)^2 - \frac{1}{8} M_{x,\alpha}^4 \frac{\hbar^{4\alpha}}{m^{3\beta}} \frac{c^{4\alpha}}{c^{6\beta}} \frac{1}{\hbar^4} (S^{(\alpha)})^4 \right] + \hbar^{\alpha-1} \frac{\partial^\alpha}{\partial t^\alpha} \frac{\partial^\alpha}{\partial x^\alpha} S(x, t). \quad (24)$$

Defining the fractional moment as

$$p_\alpha = \hbar^{\alpha-1} \frac{\partial^\alpha S}{\partial x^\alpha}, \quad (25)$$

noting that in the lowest order in α

$$\frac{d^\alpha p_\alpha}{dt^\alpha} = \frac{\partial^\alpha p_\alpha}{\partial x^\alpha} \left(\frac{dx}{dt} \right)^\alpha + \frac{\partial^\alpha p_\alpha}{\partial t^\alpha}, \quad (26)$$

with a similar the definition of the fractional velocity [10], that relates it to a fractional linear momentum,

$$v_\alpha = \left(\frac{dx}{dt} \right)^\alpha = \lambda_{\alpha,\beta} p_\alpha, \quad (27)$$

with $\lambda_{\alpha,\beta} = \left(M_{x,\alpha} \frac{c^\alpha}{c^\beta} \right)^{-1}$ we will have that

$$-\frac{\partial^\alpha}{\partial x^\alpha} (Q_\alpha(x, t) + V) \equiv F_\alpha. \quad (28)$$

where F_α is defined as the fractional force. The equation above gives us a Newtonian-like fractional dynamical equation, that coincide with $\frac{d^\alpha p_\alpha}{dt^\alpha}$ if $\alpha = 1$ and we do not consider the higher order term.

Defining also the fractional mechanical energy and the kinetic energy, respectively as

$$E_\alpha(x, t) = -\hbar^{\alpha-1} \frac{\partial^\alpha}{\partial t^\alpha} S(x, t), \quad (29)$$

and

$$K_\alpha(x, t) = M_{x,\alpha}^2 \frac{\hbar^{2\alpha}}{2m^\beta} \frac{c^{2\alpha}}{c^{2\beta}} \frac{1}{\hbar^2} \left(\frac{\partial^\alpha S}{\partial x^\alpha} \right)^2 - \frac{1}{8} M_{x,\alpha}^4 \frac{\hbar^{4\alpha}}{m^{3\beta}} \frac{c^{4\alpha}}{c^{6\beta}} \frac{1}{\hbar^4} (S^{(\alpha)})^4. \quad (30)$$

In terms of these and the quantum potential, we can rewrite eq. (20) as

$$E_\alpha(x, t) = K_\alpha(x, t) + Q_\alpha(x, t) + V. \quad (31)$$

It is important to note that if we make $\alpha=1$, all the results are in complete accord with standard Bohmian mechanics theory with the inclusion of lower relativistic correction terms.

Another point to highlight concerns energy conservation. If we assume for the phase S a dependence like a power of time,

$$S(x, t) = E \cdot \hbar(f - t^\alpha), \quad (32)$$

where E is a multiplicative constant and f is some functions depending explicitly only on x , then we obtain for the fractional energy

$$E_\alpha(x, t) = -\hbar^{\alpha-1} \frac{\partial^\alpha}{\partial t^\alpha} S(x, t) = E \hbar^\alpha \Gamma(\alpha + 1), \quad (33)$$

that is a constant. The the fractional energy can be conserved by an appropriate choice of phase.

De Broglie relations

If we write for the phase S a dependence like a power of time,

$$S(x, t) = (kx)^\alpha \pm (\omega t)^\alpha, \quad (34)$$

we will have for the energy

$$E_\alpha(x, t) = \hbar^\alpha \Gamma(\alpha + 1) (\omega)^\alpha \quad (35)$$

Note that when $\alpha = 1$, then $E = \hbar\omega$.

Inserting these phase S into the eq. (30) leads to

$$\begin{aligned} K_\alpha(x, t) &= M_{x,\alpha}^2 \frac{\hbar^{2\alpha}}{2m^\alpha} \frac{1}{\hbar^2} (\Gamma(\alpha+1)k^\alpha)^2 - \frac{1}{8} M_{x,\alpha}^4 \frac{\hbar^{4\alpha}}{m^{3\alpha}} \frac{1}{c^{2\alpha}} \frac{1}{\hbar^4} (\Gamma(\alpha+1)k^\alpha)^4 = \\ &= M_{x,\alpha}^2 \frac{1}{2m^\alpha} \frac{1}{\hbar^2} [\Gamma(\alpha+1)\hbar^\alpha k^\alpha]^2 + \mathcal{O}(p^4). \end{aligned} \quad (36)$$

Defining

$$p_\alpha = M_{x,\alpha} \Gamma(\alpha+1) \hbar^\alpha k^\alpha, \quad (37)$$

which reduces to de Broglie relations of ordinary quantum mechanics whenever $\alpha = 1$.

6 Concluding Comments

There has been considerable interest over the past recent years in the so-called theory of "weak" quantum measurements, whose aim seems to be to measure the average value of a quantum observable while negligibly disturbing the measured system [50, 51, 52, 53, 54]. Very recently, experimental observation of trajectories of a photon in a double-slit interferometer was reported, which displayed the qualitative features predicted in the de Broglie-Bohm interpretation [55, 56].

Possibilities like connections with a quantum gravity theory emerges from the fact that an modified fractional Newtonian equation could be connected with a fractional Newtonian dynamics similar to MOND of Mordehai Milgrom [57]. The natural emergence of a fractional Newtonian equation implicitly involves a non-local theory leading to a Newtonian law with memory, a characteristic of fractional derivatives. Also, the fractional energy reinforces the expectation of the presence of quantum effects. Those effects can be also associated with collective behavior in a fractal space-time tissue, where fluctuations can give rise to excitations like tisons or fractons.

Also, a version of fractional de Broglie relations naturally comes out from our equations and we recover the integer relations in the convenient limit. In connection with the probability conservation, in the fractional case, we have worked out, to the lowest order in the relativistic correction, the fractional probability current. The probability can be conserved in this non-differentiable space-time if we consider a fractional version of continuity equation that reduces to the standard one in the integer limit or, in other words, integer dimensions. As an outlook for a forthcoming work, solutions with the Mittag-Leffler instead of exponential solutions, shall be analyzed in two possibilities: non-differentiable space of solutions and coarse-grained space-time in the argument of refereed special solution function.

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